Localization and delocalization in the quantum kicked prime number rotator

Tao Ma
*Department of Modern Physics, University of Science and Technology of China, Hefei, PRC**
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The quantum kicked prime number rotator (QKPR) is defined as the rotator whose energy levels are prime numbers. The long time behavior is decided by the kick period τ and kick strength k. When $\frac{\tau}{2\pi}$ is irrational, QKPR is localized because of the equidistribution theorem. When $\frac{\tau}{2\pi}$ is rational, QKPR is localized for small k, because the system seems like a generalized kicked dimer model. We argue for rational $\frac{\tau}{2\pi}$ QKPR delocalizes for large k.

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The kicked prime number rotator is defined as

$$H = H_0 - V \sum_{n=1}^{\infty} \delta(t - n\tau), \tag{1}$$

where H_0 is the unperturbed Hamiltonian, and V is the perturbation. H_0 is a diagonal matrix. The m-th eigenvalue E_m corresponding to m-th eigenstate $|m\rangle$ of H_0 is the $|m|$-th prime number $p_{|m|}$. When $m < 0$, $E_m = E_{-m}$. $E_0 = 0$. The diagonal of H_0 is $\{\ldots, 11, 7, 5, 3, 2, 0, 2, 3, 5, 7, 11, \ldots\}$.

V is defined as

$$V = \begin{pmatrix} \cdots & \cdots & & & & \\ \cdots & 0 & k/2 & & & \\ & k/2 & 0 & k/2 & & \\ & & k/2 & 0 & k/2 & \\ & & & k/2 & 0 & \cdots \\ & & & & \cdots & \cdots \end{pmatrix}. \tag{2}$$

The Floquet operator is $F = e^{-\frac{i}{\hbar}V(\theta)} e^{-\frac{i}{\hbar}H_0\tau}$. The matrix elements of F is $F_{nm} = \exp(\frac{-i\tau E_m}{\hbar}) i^{m-n} J_{n-m}(\frac{k}{\hbar})$, where $E_m = p_{|m|}$, J_{n-m} is the Bessel function of the first kind. We set $\hbar = 1$. $F_{nm} = \exp(-i\tau E_m) i^{m-n} J_{n-m}(k)$. The system is very like the quantum kicked rotator (QKR), except its energy levels are now prime numbers.

It seems there is no classical correspondence of QKPR. Experimental implementation of such a model also seems impossible. Nevertheless it still has some theoretical interests. In the paper, we numerically calculate the evolution of QKPR. We are interested in the same problem in QKR. If the particle is in the ground state $|0\rangle$ initially, will it diffuse away in the future?

The evolution of the system is calculated by the iterative unitary matrix multiply method [1]. $F^4 = (F^2)^2$. $F^4 = (F^2)^2$. $F^8 = (F^4)^2$. And so on. In this way, we can calculate $F^{(2^{50})}$ by 50 matrix multiplies. In all our calculation, N indicates at time $2^N\tau$. For example, the first figure $N = 10$ means at time $2^{10}\tau = 1024\tau$. n is the n-th basis $|n\rangle$ and c_n is the base-10 logarithm of the absolute value of the wave function on the $|n\rangle$. n runs from -500 to 500 in our calculation.

First, we choose $k = 1$, $\tau = 2\pi\frac{\sqrt{5}-1}{2}$. The result is displayed in FIG. 1. QKPR is localized perfectly. In our simulation, the exponentially fall of the wave function never changes from $N = 8$ to $N = 50$. The wave function on the $|n\rangle$ is 10^{c_n}. From $N = 1$ to $N = 7$, the wave function is somewhat curved. After the first kick, the wave function is the 0-th column of the Floquet matrix F. The absolute value of F_{n0} is $|J_n(k)|$. $|J_n(k)|$ falls to zero faster than exponentially . This is the reason the curved form of the wave function.

Second, we choose $k = 5$, $\tau = 2\pi\frac{\sqrt{5}-1}{2}$. The result is displayed in FIG. 3. The wave function is also localized. This is expected. The sequence $\{-p_n\frac{\tau}{2\pi}Mod1\}$ is equidistributed between $[0, 1]$, when $\frac{\tau}{2\pi}$ is irrational. We denote the sequence $\{-p_n\frac{\sqrt{5}-1}{2}Mod1\}$ as QKPR$_G$. In QKR, the sequence $\{-\frac{n^2}{2}\frac{\tau}{2\pi}Mod1\}$ is also equidistributed between $[0, 1]$, for an irrational $\frac{\tau}{2\pi}$. We denote the sequence $\{-\frac{n^2}{2}\frac{\sqrt{5}-1}{2}Mod1\}$ as (QKR$_G$). We can also use the inverse Cayley transform method to convert the Floquet eigenstate equation $F\varphi = \lambda\varphi$ into an equation like Anderson localization problem. From Fishman *et al*'s argument [2], QKPR will localize.

In the left of FIG. 2, QKPR$_G$ and QKR$_G$ are displayed. Though there are apparently some correlations in QKPR$_G$ and QKR$_G$ and the correlation is different between both sequences. The correlation is surely not strong enough to destroy localization. If a sequence is periodic with a period q, then the discrete Fourier transform of the sequence is composed by q modes. To find whether there is some periodicity in the sequence, we perform a discrete Fourier transform (DFT) on the sequence. DFT of a sequence s_n of length L is defined as $F_j = \sum_{n=1}^{L} s_n e^{-i2\pi(n-1)(j-1)/L}$, where j runs from 1 to L. There are some other definitions of DFT with nuanced difference with our definition. But the difference is irrelevant to our discussion here. In the right of FIG. 2, the DFTs of both sequences are displayed. There are no rigorous periodicity in both sequences. F_k of QKPR$_G$ seems to have a trend to cluster together. Also it is less uniformly distributed than the F_j of QKR$_G$ and tends to be small.

If $\frac{\tau}{2\pi} = \frac{1}{3}$, does QKPR localize? At first thought, this seems to be a resonant case in QKR and the rotator will delocalize. The calculation result is in fact it still

*Electronic address: taomascience@gmail.com

localizes for small k. In FIG. 4, we choose $k = 1$ and $\tau = 2\pi\frac{1}{3}$ and in FIG. 5, $k = 5$ and $\tau = 2\pi\frac{1}{3}$. QKPR of $k = 1$ is apparently localized.

The explanation of the localization is QKPR with $\frac{\tau}{2\pi} = \frac{1}{3}$ and $k = 1$ is a kicked pseudo dimer rotator. The dimer model is defined as every diagonal matrix element is a probability variable which only takes two values [3, 4]. The kicked dimer model can be defined as every diagonal matrix element of $e^{-iH_0\tau}$ is a random variable which takes two values. If it takes more than two values, it is a generalized kicked dimer model. For $q = 5$, the sequence $-p_n\frac{1}{q}Mod1$ mainly takes four values. So it is a generalized kicked pseudo dimer model. The sequence $-p_n\tau Mod1$ (QKPR$_3$) is not really random. But the pseudorandomness is enough to result in localization [2]. To measure how random QKPR$_3$ is, we perform a DFT on it. In the left of FIG. 4, we compare QKPR$_3$ with a dimer sequence D$_3$, which is defined as $-\frac{1}{3}Random(n)Mod1$, where every $Random(n)$ is a random variable which takes two values 1 and 2. The F_ks of D$_3$ and QKPR$_3$ are quite close with each other, except QKPR$_3$ tends to cluster together.

Does QKPR localize for $\frac{\tau}{2\pi} = \frac{1}{3}$ and $k = 5$? We think it delocalizes. There are a series of plateaux in the wave function of QKPR. The wave function falls abruptly when approaching the boundary (cliff) of a plateau. Some plateaux disappear at $N = 14, 27, 45$. QKPR wave pass through the cliff, so it disappears intermittently.

The most obvious cliff is from $n = 140$ to 160 in FIG.6. QKPR$_3$ from $n = 139$ to 161 is

$$\begin{aligned}\{&\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\\ &\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3}\}.\end{aligned} \tag{3}$$

Note it is periodic from $n = 140$ to 160. In a perfect periodic potential the wave will always diffuse away. The quantum wave can not stay at a period potential very long. Once the wave has propagated into the phase space between $n = 140$ and 160, it diffuses away quickly. While once the wave propagate the phase space whose neighborhood $-p_n/3Mod1$ is irregular, the quantum wave is localized there at least temporarily. When $k = 1$, n from 140 to 160 is a plateau in FIG. 2. There is a transition from a plateau to a cliff when k becomes large.

Note in FIG. 4, for lots of k, Lots of F_k of QKPR$_3$ and QKPR$_G$ tend to be small. From Plancherel's theorem, there must be some F_k tends to very large. So there is some weak periodicity in the sequence QKPR$_3$. The sequence from $n = 140$ to 160 increases the periodicity of QKPR$_3$.

At $N = 54$, there is a plateau from -50 to 50 or so. Even at $N = 50$, there is a plateau from -50 to 50. The localization length of QKR is $\frac{k^2}{4} = 6.25$ or so [5, 6, 7]. So the distribution length of QKPR is much larger than the localization length. Even the result of $N = 54$ is untrustworthy, we think the quantum wave is absolutely not localized in the localization length. From $n = -150$ to 150 or so, there is apparently unneglectable quantum wave at every n at for example $N = 40$. At $N = 6$, the quantum wave has already propagated into $n \gg 6\frac{1}{4}$. QKPR with $\frac{\tau}{3} = \frac{1}{3}$ and $k = 5$ is not localized. In [8], we point out when the kick strength is larger than π, the inverse Cayley transform method breaks down. QKPR is apparently an evidence to the failure of the inverse Cayley transform method when k is large.

In this paper, we apply the iterative unitary matrix multiply method to quantum kicked prime number rotator. If $\frac{\tau}{2\pi}$ is irrational, the rotator localizes. If $\frac{\tau}{2\pi}$ is rational, for small kick strength k, the rotator localizes. As k increases, we argue there is a localization-delocalization transition.

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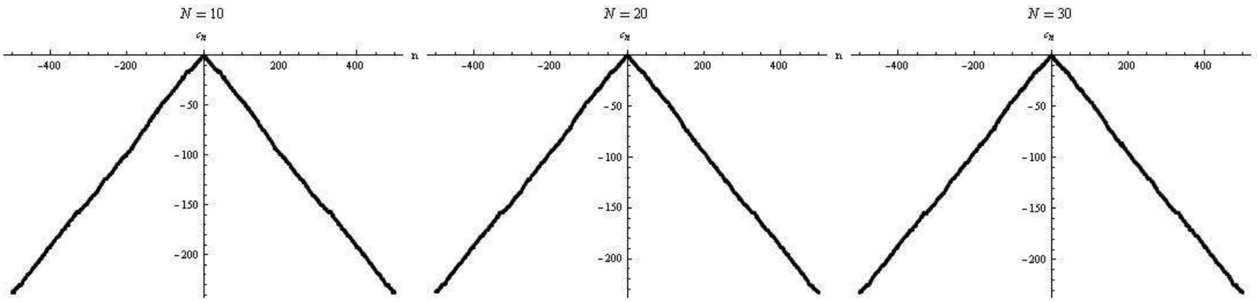

FIG. 1: QKPR wave function at different time for $k = 1$, $\tau = 2\pi\frac{\sqrt{5}-1}{2}$.

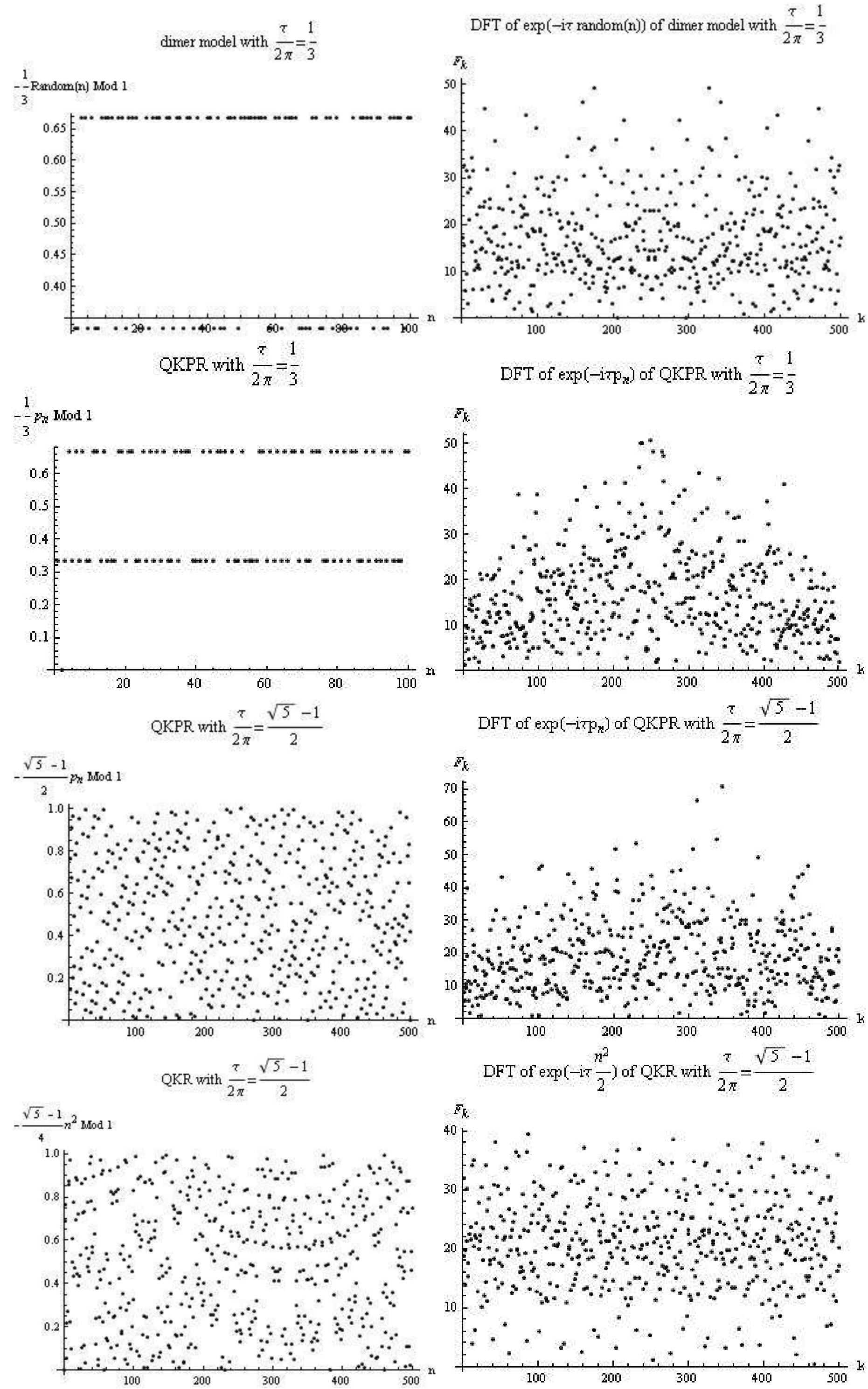

FIG. 2: Left: $-E_n\tau$ modulo 1, where E_n is the n-th energy level. Right: DFT of the sequence $\{exp(-iE_n\tau)\}$, where n runs from 1 to 500.

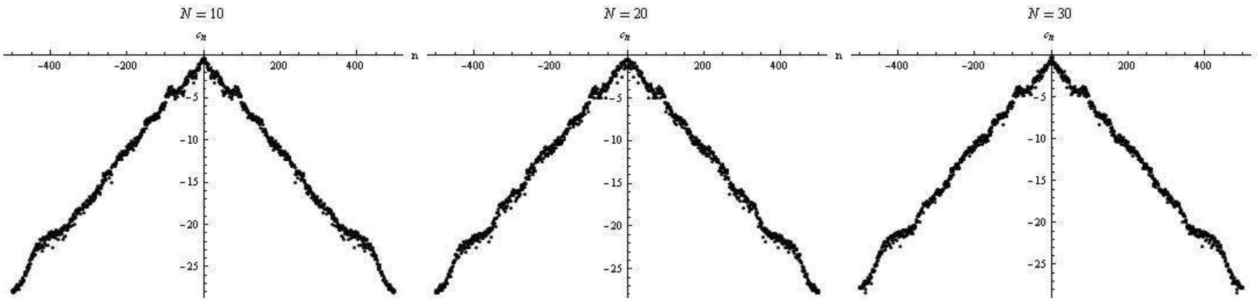

FIG. 3: QKPR wave function at different time for $k = 5$, $\tau = 2\pi\frac{\sqrt{5}-1}{2}$.

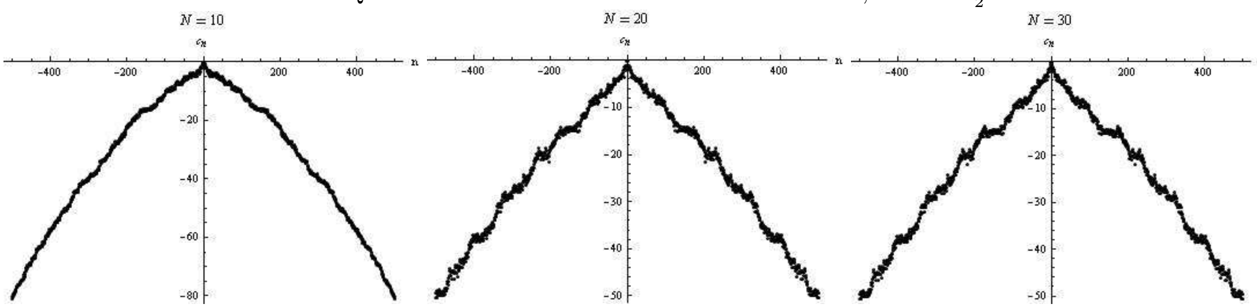

FIG. 4: QKPR wave function at different time for $k = 1$, $\tau = 2\pi\frac{1}{3}$.

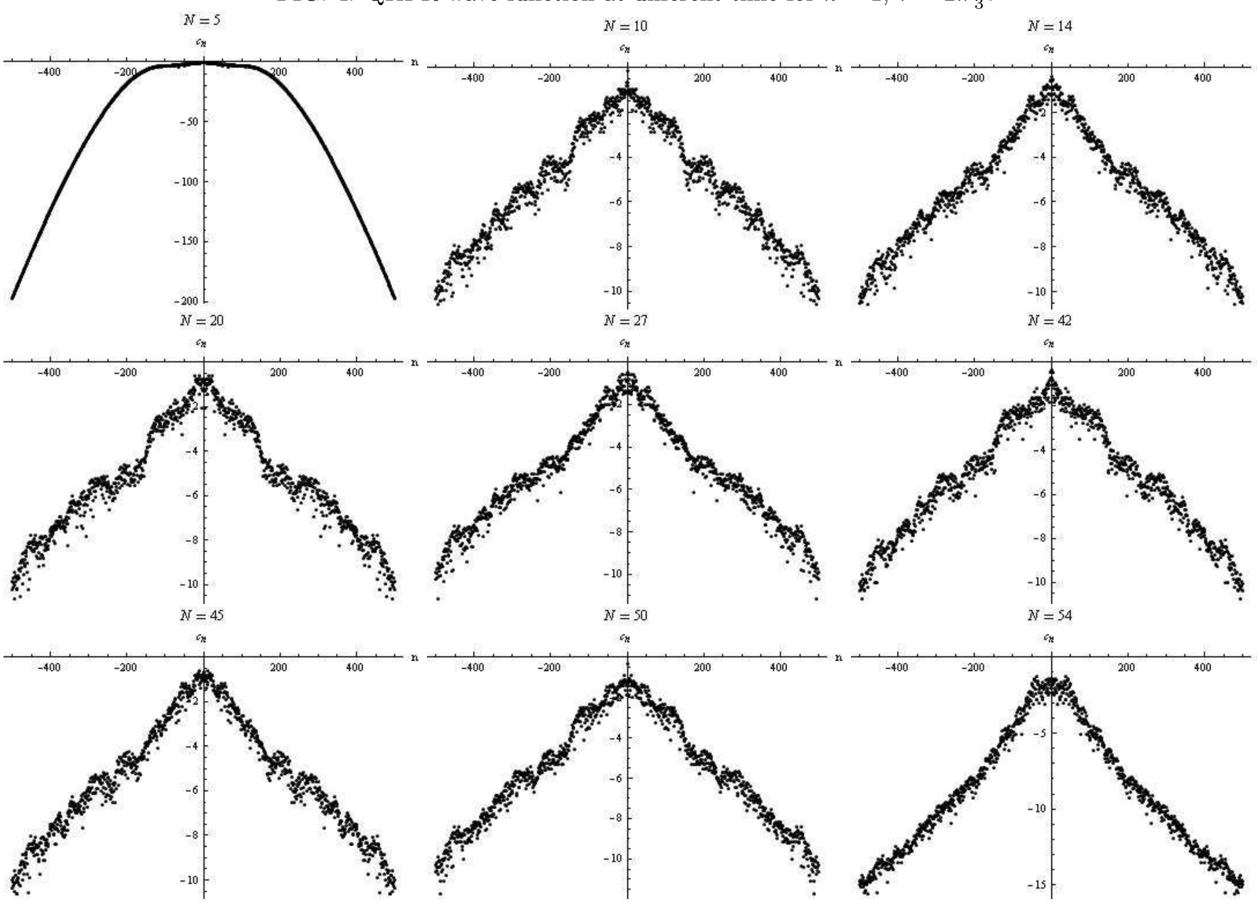

FIG. 5: QKPR wave function at different time for $k = 5$, $\tau = 2\pi\frac{1}{3}$.

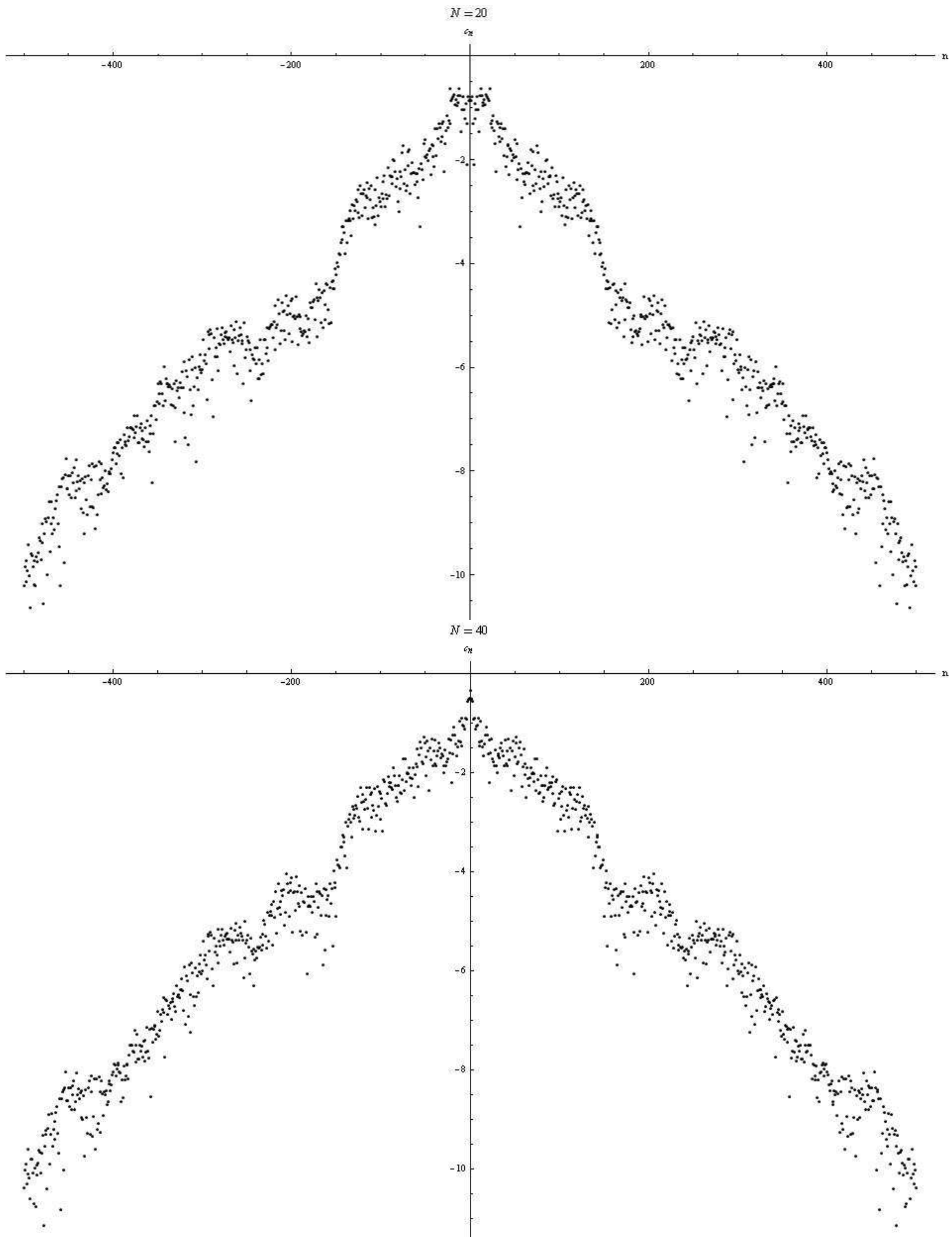

FIG. 6: QKPR wave function plateaux and cliffs at $N = 20$ and $N = 40$ for $k = 5$, $\tau = 2\pi\frac{1}{3}$.